Active clusters in disordered systems

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We introduce an exact algorithm to calculate the distribution of large low energy clusters (droplets) in disordered manifolds and disordered magnets, and we analyze the extent to which these clusters can be treated as independent two-level systems. We show that interfaces in randomly diluted networks always have broad droplet distributions, while diluted antiferromagnets in a field can have either power law or exponential droplet distributions. [S1063-651X(99)08110-6]

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Analysis of many-body systems in the presence of quenched disorder remains one of the main challenges of statistical mechanics [1]. In many cases disorder is relevant, so that ground state properties are expected to be typical. Due to a lack of self-averaging, metastability and the inability of traditional numerical methods (e.g., Monte Carlo) to navigate the rough low energy landscape, understanding of these ground state problems has often relied upon droplet scaling arguments [2] and the replica trick [3]. Though important, these methods contain unproven assumptions and/or questionable approximations and can lead to contradictory predictions. Motivated by this situation we and others are relying more heavily on new numerical methods for quenched disordered systems. In particular, the ground state of systems with quenched disorder may sometimes be mapped onto computationally fast combinatorial optimization problems [4]. Particularly fruitful are the relation of bipartite matching to rigidity and connectivity percolation [5]; the application of min-cut/max-flow to random manifolds [6] and random magnets [7,8]; and the use of min-cost flow and matching algorithms in analyzing flux lines in random media [9]. In this paper we extend the min-cut/max-flow method to find the *exact low lying cluster excitations* of random manifolds and diluted antiferromagnets in a field.

As is well appreciated, low lying droplet excitations exist in most disordered systems [2,10,11], including random magnets and disordered manifolds. A rich phenomenology has developed based on these ideas, along with experiments which support this general picture [12]. Nevertheless, theoretical understanding remains ambiguous and controversial due to a lack of models in which these excitations may be calculated convincingly. Here we study a class of models for which the density and structure of large active clusters can be calculated precisely. To enable this analysis we develop an algorithm to *exactly* calculate the ground state degeneracy of diluted antiferromagnets in a field (DAFF) and also manifolds in diluted networks. The ground state degeneracy of these ''ideal'' systems is due to a large number of regions which may be flipped without energy cost [8]. Even more importantly, our algorithm identifies those regions which may be flipped independently and so may be treated as independent two-level systems. We call these regions *active clusters* and we calculate their probability. In addition, our algorithm shows that the active clusters are frequently made up of *subclusters* which can only be flipped in a correlated way.

In experimental systems there are small perturbations which break the degeneracy of the model ground state and elevate the majority of the active clusters and subclusters out of the ground state. Moreover, if the perturbations are sufficiently small, the lowest lying excitations are the active clusters and subclusters found in the original degenerate ground state. In addition, dynamical relaxation at low temperatures is controlled by the energy barriers of the droplets (''droplets'' include the cluster excitations, and the correlated subcluster excitations), which are large compared to the differences in energy caused by small perturbations. Thus the low temperature dynamics is dominated by the distribution of active clusters and subclusters in the model system. We calculate the probability that droplets of ''s'' spins occur in random manifolds and in the DAFF on a bcc lattice, and then discuss the effect of these droplets on DAFF dynamics at low temperatures.

We illustrate the algorithm and the idea of active clusters using interfaces in the random bond Ising model with Hamiltonian

$$H_{RBIM} = -\sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j . \tag{1}$$

J_{ij} is positive and random, and its statistics are drawn from the probability distribution $P(J_{ij}) = p\,\delta(J_{ij}-J) + (1-p)\,\delta(J_{ij})$. We consider square lattices in which an interface in the {1,0} orientation is imposed by fixing the spins on the two opposite surfaces of the lattice. We then apply the method to the diluted antiferromagnet in a field (DAFF) with the Hamiltonian,

$$H_{DAFF} = J\sum_{\langle ij \rangle} \epsilon_i\epsilon_j\sigma_i\sigma_j - h\sum_i \epsilon_i\sigma_i , \tag{2}$$

on a body-centered-cubic (bcc) lattice, which is of direct relevance to experiments on $Fe_xZn_{1-x}F_2$ [12].

Our algorithm builds upon the connection between the ground state of several important random systems and the

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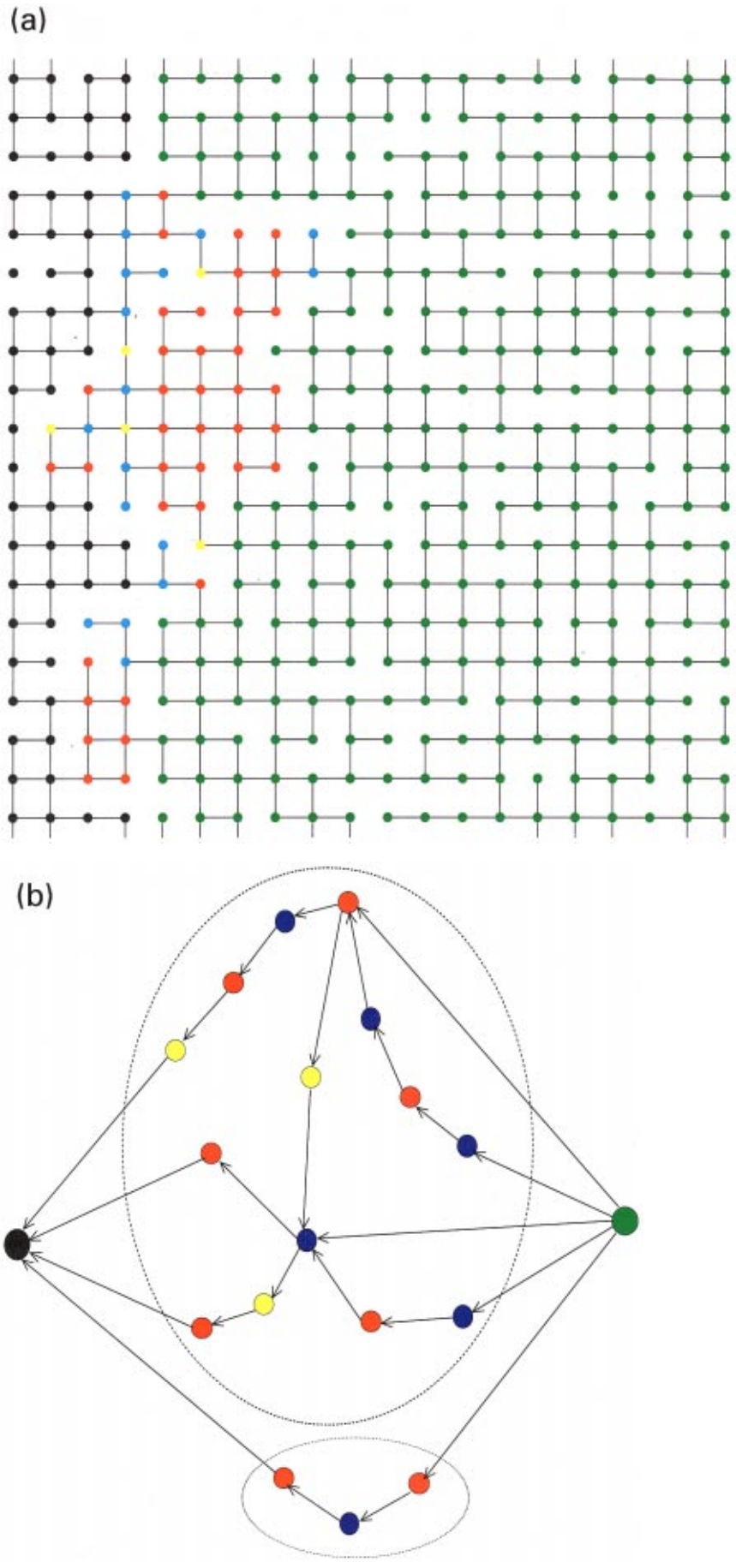

FIG. 1. (Color) (a) The interface morphology in a diluted random bond Ising model. The colored subclusters may be on either the left hand side or the right hand side of an interface without changing the interface energy. (b) A reduced graph formed from (a). In this example, only two ''clusters'' exist (inscribed by dotted lines), but there are 17 subclusters. Each directed cut (i.e., all arrows cross the cut in the same direction) is a valid minimum energy interface.

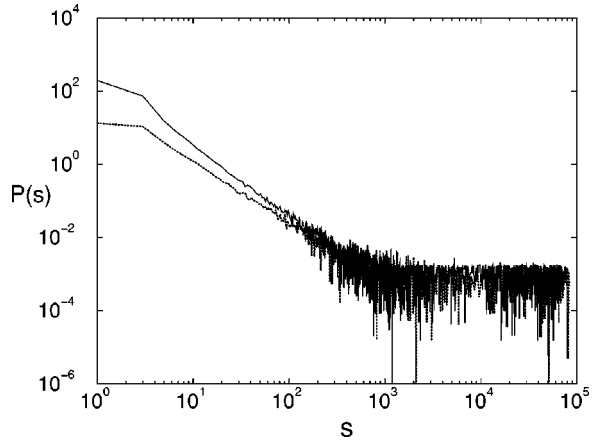

FIG. 2. The distribution of active clusters (dotted line) and of subclusters (solid line) for interfaces in a bond-diluted square lattice Ising model at $p=0.8$. The sample size is 400×400, and the distribution was found from accumulating data on 1600 configurations.

min-cut/max-flow algorithm of combinatorial optimization [6–8,4]. The min-cut/max-flow problem consists of finding the maximum load which can be pushed through a network whose bonds each have a maximum capacity c_{ij}. At each node, scalar flow is conserved. This problem has rather obvious connections with a variety of transportation and communications network problems [13]. The maximum flow is limited by a manifold of saturated bonds which occurs in the direction transverse to the direction of net flow. On this manifold, the sum of the bond capacities is minimum. For this reason it is called the minimum cut. To this point the algorithm is standard and has been extensively applied [13]. To find the ground state degeneracy we have to identify *all* minimum cuts, i.e., more than a single manifold of bonds reaches capacity at the same time. Actually, all of the bonds which reach capacity at the same time are given by the standard algorithms for min-cut/max-flow (either the augmenting path algorithm or the push-relabel algorithm) [13]. However, it is nontrivial to deduce from these bonds all the minimum cuts, i.e., the regions that the interface can move across in order to go from one degenerate cut to another [see Fig. 1(a)]. In Fig. 1(a) there are regions which may be on either side of the Ising interface without changing the interface energy. We call these regions subclusters. However, not all of these subclusters can be independently moved from one side of the interface to the other without paying an energy cost (e.g., J). That is, the subclusters are in general *dependent*. Nevertheless there are groups of subclusters which are truly independent from each other and which act as two-level systems (with the two states being on one side or the other of the interface). We call these *independent regions clusters*. As we prove elsewhere [14], counting the number of degenerate interfaces in the original random bond Ising model (RBIM) interface is equivalent to counting all *directed cuts* in the reduced graph of Fig. 1(b). Although this latter counting problem is computationally hard, the reduced graph is sufficiently small to make the problem tractable for large Ising lattices. In this way we are able to find the exact ground state degeneracy [8].

In Fig. 2, we plot the droplet distribution in the ground state of the square lattice RBIM with dilution disorder. In this plot there are two distributions. The subcluster distribution is the distribution of *sizes (number of sites)* in each of the different colored regions of Fig. 1(a), *excluding* the two clusters attached to the surfaces [black and green in Fig. 1(a)]. The *cluster* distribution is also plotted in Fig. 2, and is the size of regions which can be flipped independently, but it does not include the correlated flipping of subclusters. The distribution of droplet excitations lies between these two limits. Clusters sometimes contain just one subcluster, but are usually a combination of them. It is seen from Fig. 2, that both the clusters and subcluster distributions have broad distributions [if we force a fit $P(s)\sim1/s^{y}$, then $y\sim2$] for a range of s, but then they saturate at large s. This saturation at large s is due to the fact that with finite probability, there are two or more degenerate interfaces which have *no overlap*. When such large rare fluctuations occur, a cluster of ''extensive size'' exists between the nonoverlapping degenerate interfaces. This leads to the flat tail in Fig. 2. The distribution of active clusters is thus very broad, with extreme fluctuations (extensive clusters) occurring with finite probability in large systems. This conclusion is robust to changes in the boundary conditions and it persists when we extrapolate to infinite lattices. Stated more precisely, there is a finite probability P_e, that at least one extensive cluster exists for all $p>p_c$, and in the infinite lattice limit. For example, we calculated P_e at bond concentration $p=0.8$ for sample sizes running from $L=25$ to $L=300$, with at least 1000 samples at each system size. The value of P_e saturates at the L-independent value $P_e=0.15\pm0.01$ for $p=0.80$ for all sample sizes greater than $L=200$.

The ground state we have described above is intermediate between the ultrametric hypothesis [3] and the droplet picture [2]. There are degenerate nonoverlapping ground states (though not an extensive number) [2] and there are broad distributions of active clusters [3]. Note, however, that degenerate ground states are not ''ultrametrically related'' as ''loops'' occur in any attempt to form a ''tree'' representation of the relationships between the ground states [e.g., from Fig. 1(a)].

The diluted antiferromagnet in a field (DAFF) is a key model of disordered systems, as there are direct relations to precisely controlled experimental systems such as $Fe_xZn_{1-x}F_2$ [12]. For high magnetic concentrations $x\rightarrow1$, and at high critical temperatures (which implies low critical fields) experiments indicate that the DAFF has a critical behavior consistent with the random field Ising model (RFIM), as predicted theoretically [11]. However the experiments, especially at low magnetic concentrations, (e.g., $p<0.60$) show very slow relaxation, for example, in the excess magnetization. We show that a broad distribution of active clusters exists in DAFF, and that these clusters are one source of the slow relaxation that is observed.

The DAFF is given by Eq. (2), with $J>0$ where ϵ_i are quenched independent random variables distributed according to $P(\epsilon_i)=x\delta(\epsilon_i-1)+(1-x)\delta(\epsilon_i)$ with the fraction of magnetic sites (Fe) being x. The experimental system is closely modeled by Ising spins on a site-diluted body-centered cubic lattice with the nearest neighbor exchange parameter being known quite accurately [15]. On bipartite lattices, flipping the spins on one of the sublattices maps the DAFF onto a bond diluted ferromagnetic Ising model in an

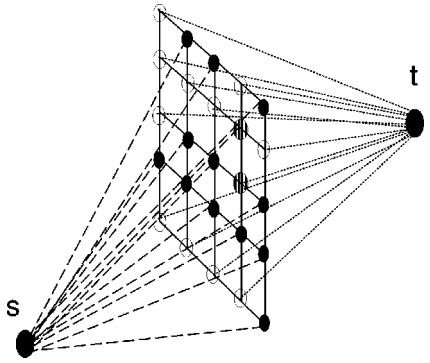

FIG. 3. A representation of how to calculate the ground state domain structure of a bimodal random field magnet using the min-cut/max-flow method. All of the positive fields are connected to the source (s), all the negative to the target (t). There is a ferromagnetic exchange J between sites, and the amplitude of the random field is $3.5J$. Different hatched, filled, and open circles indicate different spin domains. The hatched circles are a degenerate cluster.

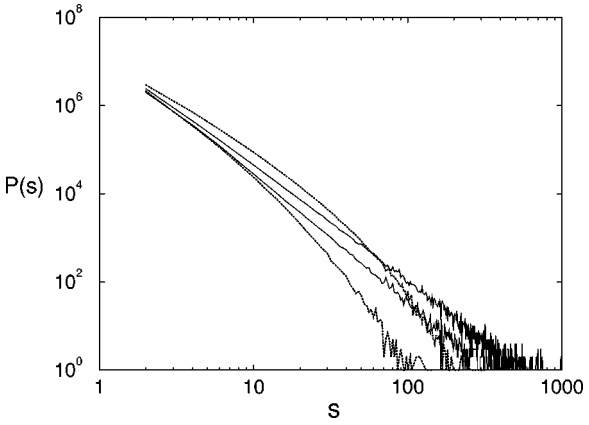

FIG. 4. The distribution of active clusters in the DAFF. From 4000 configurations of 50^3 body-centered-cubic lattices. Near the critical field ($h=0.499, p=0.31$—top solid line), and ($h=3.001, h=0.60$—bottom solid line), there is a power law distribution of active clusters. Away from the critical field there are still active clusters, but now the distribution falls off much more rapidly ($h=1.601, p=0.31$—top dotted line) and ($h=0.139, p=0.31$—bottom dashed line).

alternating field (fields are opposite on each sublattice). This is a subclass of a more general problem which is solved by the min-cut/max-flow algorithm [8]. The general problem which is solved by this algorithm is a random bond Ising model (with any range interaction, provided all couplings are ferromagnetic) in an arbitrary local field (the local fields can be either sign). One way in which min-cut/max-flow may be used to solve these problems is illustrated in Fig. 3 for a square lattice. In this figure, positive local fields are connected to the target (t), while negative local fields are connected to the source (s). The minimum cut must choose whether to break the lines to the target or source, and every time it changes its mind, a ferromagnetic bond is broken. This construction extends to bipartite lattices in three dimensions, e.g., to the body-centered cubic lattice, including the degeneracy calculation discussed above for the interface problem.

We have carried out extensive calculations of the distribution of active clusters in the DAFF on a bcc lattice, at various dilutions. In all cases (see Fig. 4), there is a power law distribution of active clusters near the critical field. The exponent of these power law decays is $y \simeq 2.8$. Well away from the critical field, the distribution of active clusters decays much more rapidly than a power law (close to exponential).

We have shown that both the random manifold problem and the DAFF problem have broad distributions of droplets at low temperatures. Both the independent two-level systems (the active clusters) and the dependent subclusters have these properties. If we consider only the independent two-level systems, the response at low temperatures is described by the droplet models of spin glasses [2]. That is, a cluster containing s spins has an activation barrier, $E_B = As^{\psi}$ [2], where $\psi < 1$ is the barrier exponent and A is a constant. The size of the ''largest thermally active cluster'' occurring during an experiment over time scale t is estimated from

$$\frac{t}{\tau_0} e^{-As_c^{\psi}/(k_B T)} \sim 1 \quad \text{so} \quad s_c \approx \left[\frac{k_B T}{A} \ln\left(\frac{t}{\tau_0} \right) \right]^{1/\psi}, \qquad (3)$$

where τ_0 is the inverse of an attempt frequency. Clusters of size $s < s_c$ are thermally active, while those of size $s \gg s_c$ remain frozen. If there is a power law of active clusters, it is easy to show that logarithmic relaxation [2] and $1/f$ noise [16,17] should be typical in experiments. The advantage of the computational methods introduced here is that we can explicitly calculate the distribution of droplets, the degree to which they interact, and their explicit structure, and hence provide detailed information about the origin of the fluctuations leading to logarithmic relaxation and $1/f$ noise in disordered systems. The detailed analysis of the consequences for experiments on diluted antiferromagnets will be discussed elsewhere.

We have presented precise calculations of the distribution of droplet excitations in two nontrivial disordered systems, namely, random manifolds and the diluted antiferromagnet in a field. In both cases, there are broad distributions of droplets (active clusters) which frequently act as independent two-level systems.

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